On the Complexity of Co-secure Dominating Set Problem

Bhawani Sankar Panda[a], Soumyashree Rana[a], Sounaka Mishra[b]

[a]Department of Mathematics, IIT Delhi, New Delhi, Delhi India, 110016
[b]Department of Mathematics, IIT Madras, Chennai, Tamil Nadu India, 600036

Let $G = (V, E)$ be a finite, simple, and undirected graph without isolated vertices. A set $D \subseteq V$ is called a *dominating set* if every vertex in $V \setminus D$ has at least one neighbor in D. The smallest size of such a set is the *domination number*, denoted by $\gamma(G)$. Given a graph G, the minimum dominating set problem (Min Dom), asks for a dominating set of minimum size. Min Dom and its variations have been widely studied due to their theoretical significance and real-life applications. For detailed survey refer to [6, 7].

A dominating set $S \subseteq V$ is called a *secure dominating set*, if for every $u \in V \setminus S$ there exists a neighboring vertex $v \in S$, of u such that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set. This variation of domination was introduced by Cockayne et al. [4]. This problem along with its many variants has attracted considerable attention.

A related concept is the co-secure dominating set (CSDS). A dominating set $S \subseteq V$ is a *co-secure dominating set* (CSDS) if for each vertex $u \in S$ there exists a neighboring vertex $v \in V \setminus S$, of u such that $(S \setminus \{u\}) \cup \{v\}$ is a dominating set. The minimum size of such a set is the *co-secure domination number*, denoted by $\gamma_{cs}(G)$. Given a graph $G = (V, E)$, the minimum co-secure dominating set problem (Min Co-secure Dom), asks for a co-secure dominating set of minimum size. The corresponding decision problem is Co-secure Dominating Set.

Secure and co-secure domination have important applications in networks that require both robustness and adaptability, such as communication, sensor, and distributed systems. They model scenarios where resources (e.g., servers, guards, or facilities) can be reassigned or replaced without losing overall coverage or functionality. These concepts are particularly useful in designing fault-tolerant systems, surveillance strategies, and dynamic service networks.

Min Co-secure Dom was introduced by Arumugam et al. [2], where they showed that Co-secure Dominating Set is NP-complete for bipartite, chordal, and planar graphs. They also determined γ_{cs} for some families of the standard graph classes such as paths, cycles, wheels, and complete t-partite graphs. Later Zou et al. [11] proved that the co-secure domination number of proper interval graphs can be computed in linear time. Kusum and Pandey [8] proved that Min Co-secure Dom is NP-hard to approximate within a factor of $(1 - \varepsilon) \ln |V|$ for any $\varepsilon > 0$ and in terms of maximum degree Δ, it can be approximated within a factor $(\Delta + 1)$. Moreover, it is APX-complete for graphs with maximum degree 4.

Despite this, there are still gaps in understanding this problem from algorithmic and structural point of view. In this work, we address some of these and extend the study of Min Co-secure Dom from algorithmic perspective by using certain properties of minimum double dominating set under some assumptions. The main contributions of this work are summarized below.

First, we prove that the known approximation hardness result [8] still holds even if the input graph has some extra "nice" conditions, such as the input graph is a perfect elimination bipartite graph or a star convex bipartite graph. This improves the result of Kusum and Pandey [8]. To obtain our lower bound, we establish an approximation preserving reduction from Min Dom to Min Co-secure Dom. We need the following lower bound result on Min Dom.

Theorem 1. *([3, 5]) Unless* P=NP, Min Dom *can not be approximated within a factor of* $(1 - \varepsilon) \ln |V|$*, for any* $\varepsilon > 0$*. Such a result holds for* Min Dom *even when restricted to bipartite graphs.*

By using this theorem, we will prove similar lower bound results for Min Co-secure Dom for two subclasses of bipartite graphs, namely perfect elimination bipartite graphs and star convex bipartite graphs.

Theorem 2. *Unless* P=NP, Min Co-secure Dom *can not be approximated within* $(1 - \varepsilon) \ln |V|$*, for any* $\varepsilon > 0$*, even for perfect elimination bipartite graphs and star convex bipartite graphs.*

Next, we propose an approximation algorithm for MIN CO-SECURE DOM whose approximation ratio is a logarithmic factor of the number of vertices of the input graph. To obtain the desired ratio, we require the approximation ratio of the minimum double dominating set problem (MIN DOUBLE DOM). Given a graph $G = (V, E)$, the MIN DOUBLE DOM aims to find a set $D \subseteq V$ of minimum cardinality such that $|N(v) \cap D| \geq 2$, for all $v \in V \setminus D$. We shall denote $\gamma_2(G)$ as the cardinality of a minimum double dominating set of G. We will use the following proposition and a few lemmas to analyze our approximation algorithms' performance.

Lemma 1. *If G is a connected graph with at least 3 vertices then every minimal double dominating set D_2 of G is a proper subset of V. Moreover, if $\delta(G) \geq 2$ then every minimal double dominating set D_2 is a co-secure dominating set of G.*

We further prove that from a CSDS D of G, one can construct in polynomial time, a double dominating set D_2 of G of size at most twice the size of D.

Lemma 2. *Let G be a graph with $\delta(G) \geq 2$. For any* CSDS *D of G, in polynomial time one can construct a double dominating set $\overline{D}$ of G, such that $|\overline{D}| \leq 2|D|$.*

By using the above lemma, we prove bounds on $\gamma_2(G)$, which we will use in designing approximation algorithms for MIN CO-SECURE DOM.

Lemma 3. *For every graph G with $\delta(G) \geq 2$, $\gamma_{cs}(G) \leq \gamma_2(G) \leq 2\gamma_{cs}(G)$. Moreover, these bounds are tight.*

Theorem 3. MIN DOUBLE DOM *can be approximated with an approximation ratio of $O(\ln |V|)$, where V is the vertex set of the input graph G. It can also be approximated within a factor of $1 + \ln(\triangle + 2)$, where $\triangle$ is the maximum degree of G.*

Now, we have the following theorem to compute an approximate solution of MIN CO-SECURE DOM using Theorem 3 and Lemma 3. This result improves the existing approximation $(\Delta + 1)$ [1].

Theorem 4. MIN CO-SECURE DOM *can be approximated within a factor of $O(\ln |V|)$, for graphs with $\delta(G) \geq 2$. It can also be approximated within a factor of $2 + 2\ln(\triangle + 2)$, where $\triangle$ is the maximum degree of G.*

Further, we study the problem on bounded degree graphs. For a 3-regular graphs, we present a $\frac{8}{3}$-approximation algorithm and For 4-regular graphs, a $\frac{10}{3}$-approximation.

Lemma 4. MIN CO-SECURE DOM *is approximable within a factor of $\frac{8}{3}$ and $\frac{10}{3}$ for 3-regular graphs and 4-regular graphs, respectively.*

Furthermore, we show that MIN CO-SECURE DOM is APX-complete for 3-regular graphs, strengthening the existing APX-completeness result for maximum degree 4 graphs. Note that the class APX is the set of all optimization problems which admit a c-approximation algorithm, where c is a constant. From Theorem 4 it follows that MIN CO-SECURE DOM can be approximated within a factor of 5.583 for graphs with maximum degree 4. We improve this approximation factor to $\frac{10}{3}$ (Lemma 4).

The APX-completeness of MIN CO-SECURE DOM for 3-regular graphs follows from a connection to the Minimum Partial Monopoly problem, which we recall below.

Definition 1 ([10]). (MIN PARTIAL MONOPOLY) *Given a graph $G = (V, E)$, minimum partial monopoly problem is to find a set $M \subseteq V$ of minimum cardinality such that for each $v \in V \setminus M$, $|M \cap N[v]| \geq \frac{1}{2}|N[v]|$.*

It is known that for 3-regular graphs, MIN PARTIAL MONOPOLY is APX-complete [9]. It is easy to observe the following lemma.

Lemma 5. *Let G be a 3-regular graph. A partial monopoly set M of G is a double dominating set of G and vice versa.*

Now, we prove that MIN CO-SECURE DOM is APX-complete for 3-regular graphs by establishing a reduction from MIN PARTIAL MONOPOLY for the corresponding graphs.

Theorem 5. MIN CO-SECURE DOM *is* APX-*complete for 3-regular graphs.*

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